Optimal design of two-stage speed reducer

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Abstract: In this paper an optimal design of two-stage speed reducer is presented. The novelty of this article consists in the complex and complete approach of the optimal design of gearings. The chosen objective function was the volume bounded by the inner surface of the reducer housing. For this example of optimal design, eleven genes were taken into consideration and a set of thirty six constraints were formulated. In solving the optimization problem we used an original dual-phase evolutionary algorithm (DPEA) inspired from the evolutionary concept of "punctuated equilibrium". DPEA is implemented in Cambrian v.3.2 which is in operation at the Optimal Design Centre of the Technical University of Cluj-Napoca.

Key-Word: Evolutionary algorithms, Optimal design, Helical gears.

1 Introduction
The main goal of this paper lies in emphasizing once again the advantages of the optimal design of all sorts of products as compared to the classical design. In this particular case we deal with the optimal design of a two stages speed reducer. A two-stage speed reducer optimization problem induces a number of challenges especially when the design problem involves gear kinematics, geometry and strength. The resulting optimization problem involves design variables which can be integer (number of teeth), discrete (normal module), and real (gear width). Many researchers have reported solutions to this problem. In the chronological order the first who dealt with this problem was Osman [8], in 1978. He made a design synthesis of a nine-speed gear drive. The objective of the synthesis was to minimize the size of all gears from the mesh and speed ratio equations so that the size of the largest gears is kept to a minimum. Due to the mesh and speed ratio equations, it is found that only the following three independent parameters need to be selected: \( x_1 \), \( x_2 \) and \( x_3 \) (\( x_1 \)-gear ratio, \( x_2 \), \( x_3 \) – constraints). Because of practical considerations, the minimization of \( |x_2 - x_3| \) was found to result in the reduction of the cost of manufacturing the gear drive. In [1], Aberšek described an expert system (STATEX) to design and manufacture a gear box. In the first stage of the process, genetic algorithms were used to determine the optimal dimensions of a gear box (with special requirements) then, the expert system took technological requirements into account, related to the selection of cutting tool and cutting conditions, the special sequence of machining, the tolerances etc. Other researchers that reported solutions to optimal design of one stage speed reducer problem were Kuang et al. in [3] and Liand in [5]. However, the solutions reported in [3] and [5] are non-feasible. In [6], [9] and [10] an optimal design problem of a one stage speed reducer is presented. In these papers the objective was to minimize the weight of the reducer. In [6], [9] and [10] the authors used a set of seven variables as follows: face width, module of teeth, number on pinion teeth, length of input shaft between the bearings, length of output shaft between bearings, diameter of shaft 1 and diameter of shaft 2. The objective function was subjected to a simply formulated (from a mechanical point of view) set of eleven constraints. Mezura in [6] utilized this problem only to test one of previous version of his Simple Multimembered Evolution Strategy (SMES) software. Vu in [7] presented a study of the optimization and regression techniques for optimum determination of partial ratios of three-step helical gearboxes in order to minimize different objectives (gearbox length, cross section dimension or gearing mass). The simplification made in this approach could affect the optimization results. In [4], Li et al presented an optimal design problem of a two-stage speed reducer. The purpose of the paper is to obtain the multi-objective optimization design scheme of a gear reducer with a Fuzzy Genetic Algorithm (FGA). However they use a set of only 7 constraints too little related to the design realities. The approach chosen by the authors for the design problem makes
it one of the most round works from all above mentioned papers. As opposed to the researchers mentioned above who were only interested in the programming and mathematical aspects of the problem, we also took into consideration the strength component utilizing in the constraints calculi DIN 3990 [11] norms. With DIN 3990 we obtained a design problem with a higher level of complexity. We dealt separately with the gearing (which in this case represents a two pairs of gears – a two-stage speed reducer) and the shafts. The separate treatment is related to our experience (as when we could find a reducer gearing optimal design solution it was easy to reach the matching shafts solution). The reducer shafts will be the topic of another paper. Thus, the objective consists in minimizing the volume defined by the reducer housing inner surface. It is known that a reducer is a part of an industrial application that has to occupy the smallest space possible; its weight is given by its housing, which is by far the heaviest part of the assembly. In the following subsections, we outline the formulation of the optimal reducer design problem in a systematic manner.

Note that for our design problem we used a set of 36 constraints and for the shafts problem a set of 61 constraints (i.e. a total of 97 constraints), while in [6], [9] and [10] there were used only 11 simplified constraints (i.e. a total of 97 constraints), while in [6], [9] and [10] there were used only 11 simplified ones. Even in the work [4], which is the most complete from all above presented, they used only a number of 7 restrictions.

2 Dual-phase enhanced evolutionary algorithm

Optimization problems with a very large number of constraints can be very difficult to solve. In order to remove this shortcoming, a dual-phase enhanced evolutionary algorithm (DPEA) inspired from the evolutionary concept of "punctuated equilibrium" is presented in this paper. The main idea behind this algorithm is its operation in two phases. In each phase, the individual’s fitness is determined by another factor. In Phase 1, the individuals’ fitness depends only on the way in which an individual is more suitable (or not) in terms of constraints (population “fight for survival” and there is no interest for the best individual). This phase is a kind of “feasible individual generator”. The algorithm moves into the second phase when the number of feasible individuals of the population exceeds a preset threshold. Phase 2 is a common evolutionary algorithm (sometimes a simple genetic algorithm). In the following we present, in short, how to determine an individual’s fitness in both phases of the algorithm. The optimization problem consists of an objective function \( f \) accompanied by certain number of constraints. The search space is considered the space of the \( n - \) dimensional decision vectors:

\[
\mathbf{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})
\]

where:

\( n \) – number of genes (variables);

The constraints of the problem are:

- \( n_u \) inequality type constraints:
  \( g_i(\mathbf{x}) \leq 0, \quad i = 1, n_u \)

- \( n_u \) strict inequality type constraints:
  \( g_i(\mathbf{x}) < 0, \quad i = n_u + 1, n_u + n_i \)

- \( n_e \) equality type constraints:
  \( g_i(\mathbf{x}) = 0, \quad i = n_u + n_u + 1, n_u + n_i + n_e \)

In order to use the use this constraints in our algorithm we needed to aggregate them in the following form:

\[
G_i(\mathbf{x}) = \begin{cases} 0, & g_i(\mathbf{x}) \leq 0 \\ g_i(\mathbf{x}) + \varepsilon, & g_i(\mathbf{x}) > 0 \end{cases}
\]

\( i = 1, n_u \)

\[
G_i(\mathbf{x}) = \begin{cases} 0, & g_i(\mathbf{x}) < 0 \\ g_i(\mathbf{x}), & g_i(\mathbf{x}) \geq 0 \end{cases}
\]

\( i = n_u + 1, n_u + n_i \)

\[
G_i(\mathbf{x}) = \begin{cases} 0, & g_i(\mathbf{x}) = 0 \\ |g_i(\mathbf{x})|, & g_i(\mathbf{x}) \neq 0 \end{cases}
\]

\( i = n_u + n_i + 1, n_u + n_i + n_e \)

where:

\( \varepsilon \) – very small positive quantity.

In each phase, for each individual a so called score is computed. The partial score of an individual \( \mathbf{x}_j \) (from those \( N \) individuals of the population \( \mathbf{x}_j, j = 1, N \), regarding to the constraint \( i, \ i = 1, n_u + n_i + n_e \) is calculated as follows:

\[
PS_i(\mathbf{x}_j) = G_i(\mathbf{x}_j) / \sum_{k=1}^{N} G_i(\mathbf{x}_k)
\]

Eventually, the (individual) score of each individual \( \mathbf{x}_j, j = 1, N \) of the population is:

\[
S(\mathbf{x}_j) = \sum_{i=1}^{n_u+n_i+n_e} PS_i(\mathbf{x}_j)
\]

Obviously, any feasible individual has null score. During Phase 1 the population is sorted by the score, and during Phase 2, the population is sorted by score and objective value. In both phases the
fitness of an individual is set according to its rank. DPEA is implemented in our Cambrian software.

3 Design problem formulation
The aim of our design problem is to obtain a two-stage speed reducer (see Fig. 1) as compact as possible in the following input data:
- Electrical engine horsepower: \( P = 2.9 \text{ kW} \);
- Overall transmission ratio: \( i = 7.6 \);
- Rotational speed of shaft 1: \( n_1 = 925 \text{ rpm} \);
- Gear necessary life: \( L_{dt,2,3,4} = 8000 \text{ hours} \);
- Basic metric rack profile – ISO53;

![Fig. 1 The reducer sketch](image)

The materials chosen for the gears are:
- Pinions: – 42CrMo4, quenched and tempered at \( HB_t = 3000 \text{ MPa} \);
  - Allowable Hertzian stress: \( \sigma_{H,lim1,3} = 760 \text{ MPa} \);
  - Allowable bending stress: \( \sigma_{F,lim1,3} = 580 \text{ MPa} \);
- Wheels:
  - 41Cr4, quenched and tempered at \( HB_t = 2700 \text{ MPa} \);
  - Allowable Hertzian stress: \( \sigma_{H,lim2,4} = 720 \text{ MPa} \);
  - Allowable bending stress: \( \sigma_{F,lim2,4} = 560 \text{ MPa} \).

4 Optimization of the two-stage speed reducer
In order to perform the optimal design of the two-stage speed reducer it is necessary to set up: the variables (genes) that uniquely describe the problem, the objective function and the problem constraints. Hereinafter, since the optimization will be performed using evolutive algorithms, instead of the term variable we will use the term gene.

4.1 Genes
The design problem genes are presented in Table 1.

<table>
<thead>
<tr>
<th>No</th>
<th>Genes</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Transmission ratio, ( i_{1,2,3,4} )</td>
<td>1.12...40</td>
</tr>
<tr>
<td>2</td>
<td>Working centre distance, ( a_w )</td>
<td>71...400</td>
</tr>
<tr>
<td>3</td>
<td>Normal addendum modification coefficient, ( x_{n1} )</td>
<td>-0.5...+1</td>
</tr>
<tr>
<td>4</td>
<td>Length width coefficient, ( \psi_{w1} )</td>
<td>0.2...0.8</td>
</tr>
<tr>
<td>5</td>
<td>Standard pitch cylinder helix angle, ( \beta_1 )</td>
<td>7...15°</td>
</tr>
<tr>
<td>6</td>
<td>Number of teeth of pinion, ( z_1 )</td>
<td>25...56</td>
</tr>
<tr>
<td>7</td>
<td>Working centre distance, ( a_w )</td>
<td>71...400</td>
</tr>
<tr>
<td>8</td>
<td>Normal addendum modification coefficient, ( x_{n3} )</td>
<td>-0.5...+1</td>
</tr>
<tr>
<td>9</td>
<td>Length width coefficient, ( \psi_{w3} )</td>
<td>0.2...0.8</td>
</tr>
<tr>
<td>10</td>
<td>Standard pitch cylinder helix angle, ( \beta_2 )</td>
<td>7...15°</td>
</tr>
<tr>
<td>11</td>
<td>Number of teeth of pinion, ( z_3 )</td>
<td>25...56</td>
</tr>
</tbody>
</table>

Note that the genes with number 1,2,6,7 and 11 (see Table 1) have only standardized values in the appropriate mentioned range.

4.2 Objective function
The expression of the volume defined by the inner surface of the reducer housing is:
\[
V_r = A_{fr} \cdot L_r
\]
where:
- \( A_{fr} \) – frontal surface area of the reducer housing (see Fig. 2), [\text{mm}^2];
- \( L_r \) – width of the housing reducer, [\text{mm}].

4.3 Constraints
The solutions of the optimization program have to satisfy the following constraints listed bellow. All values of these constraints have to be negative or maximum zero.

**C1-2.** The relative error of the actual transmission ratio has to be between -2.5% and +2.5%.

\[
g_{i2} = \begin{cases} 
40 \cdot |i_{1,2,3,4} - i_{1,2,3,4}| / i_{1,2,3,4} - 1 & \text{if } i_{1,2,3,4} < 4 \\
33.3 \cdot |i_{1,2,3,4} - i_{1,2,3,4}| / i_{1,2,3,4} - 1 & \text{if } i_{1,2,3,4} \geq 4 
\end{cases}
\]
where:

\[ t_{12,34} \] — standardized transmission ratio.

**C3-4.** The Hertz stress (\( \sigma_{H,1,2} \)) should be less or equal to the allowable Hertz stress (\( \sigma_{HP,1,2} \)) for both gearings.

\[
g_{3,4} = \frac{\sigma_{H,1,2}}{\sigma_{HP,1,2}} - 1
\]

**C5-8.** The bending (\( \sigma_{F,1,2,3,4} \)) stress at the tooth base has to be less or equal to the allowable bending stress (\( \sigma_{FP,1,2,3,4} \)) for all gears.

\[
g_{5-8} = \frac{\sigma_{F,1,2,3,4}}{\sigma_{FP,1,2,3,4}} - 1
\]

**C9-12.** The normal addendum modification coefficient should be in such range that the undercutting of all gears teeth does not get worse.

\[
g_{9-12} = \left[ 14 - 17 \cdot x_{n1,2,3,4} \right] / z_{n1,2,3,4} - 1
\]

**C13-16.** The profile shift should be in such range that the tooth thickness at top of all gears does not decrease under a certain value.

\[
g_{13-16} = c_{aw} \cdot m_{n1,2,3,4} / s_{aw1,2,3,4} - 1
\]

**C17-18.** Radial contact ratio should be greater than a certain imposed value.

\[
g_{17,18} = \epsilon_{aw1,2} / \epsilon_{a_{-1,2}} - 1
\]

**C19-20.** The normal addendum modification coefficient, \( x_{n2,4} \) should be in the range of \([-0.5...1]\).

\[
g_{19,20} = \left[ x_{n2,4} - 0.25 \right] / 0.75 - 1
\]

**C21-32.** For the span measurement, the following conditions should be satisfied:

\[
g_{21-24} = \left( w_{n1,2,3,4} \cdot \sin \beta_{n1,2} + 5 \right) / b_{1,2,3,4} - 1
\]

\[
g_{25,26} = p_{n1,3} / p_{n1,3} - 1
\]

\[
g_{27,28} = p_{n1,3} / p_{n1,3} - 1
\]

\[
g_{29,30} = p_{n2,4} / p_{n2,4} - 1
\]

\[
g_{31,32} = \rho_{n2,4} / \rho_{n2,4} - 1
\]

**C33-34.** The number of the pinion teeth \( z_{1,3} \) and the number of the wheel teeth \( z_{2,4} \) should be co-prime numbers.

\[
g_{33,34} = \begin{cases} -1 & \text{if } z_{1,3} \text{ and } z_{2,4} \text{ are coprime} \\ 1 & \text{otherwise} \end{cases}
\]

**C35.** The non-interference condition between the high-speed big gear and the output shaft.

\[
g_{35} = 0.5 \cdot d_{a2} / \left( a_{w2} - 5 \right) - 1
\]

**C36.** The level of oil inside the reducer housing should be in the range of \([H_{\text{min}}, H_{\text{max}}]\).

\[
g_{36} = 10 / \Delta H - 1
\]

where:

\( \Delta H \) — difference between the upper and lower level of oil, inside the reducer case, [mm].

\( H_{\text{max}} \) — upper level of oil, [mm];

\( H_{\text{min}} \) — lower level of oil, [mm].

### 4.4 Results

In solving this optimization problem, our own Cambrarian v.3.2 software was used. Written in Java, Cambrarian is a platform that allows the assembling of all sort of evolutive algorithms in an original manner. The obtained values of all considered genes are presented in Table 2.

<table>
<thead>
<tr>
<th>No</th>
<th>Genes corresponding to the first stage</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Transmission ratio, ( t_{12,standard} )</td>
<td>2.8</td>
</tr>
<tr>
<td>2</td>
<td>Working centre distance, ( a_{w,1} )</td>
<td>80 mm</td>
</tr>
<tr>
<td>3</td>
<td>Normal addendum modification coefficient, ( x_{n1} )</td>
<td>0.84</td>
</tr>
<tr>
<td>4</td>
<td>Length width coefficient, ( \psi_{w,1} )</td>
<td>0.49</td>
</tr>
<tr>
<td>5</td>
<td>Standard pitch cylinder helix angle, ( \beta_1 )</td>
<td>13.5°</td>
</tr>
<tr>
<td>6</td>
<td>Number of teeth of pinion, ( z_{1,1} )</td>
<td>27</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No</th>
<th>Genes corresponding to the second stage</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>Working centre distance, ( a_{w,2} )</td>
<td>100 mm</td>
</tr>
<tr>
<td>8</td>
<td>Normal addendum modification coefficient, ( x_{n3} )</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>Length width coefficient, ( \psi_{w,2} )</td>
<td>0.74</td>
</tr>
<tr>
<td>10</td>
<td>Standard pitch cylinder helix angle, ( \beta_2 )</td>
<td>12.75°</td>
</tr>
<tr>
<td>11</td>
<td>Number of teeth of pinion, ( z_{3} )</td>
<td>34</td>
</tr>
</tbody>
</table>

In Table 3, the main characteristics of the reducer gearings (classical and optimal solutions) are shown side-by-side.

### 5 Conclusions

The comparative study of the solutions shown in Table 3 leads to the following conclusions:

- The volume of the inner surface of the reducer housing calculated with the classical method is \( 12.269 \cdot 10^3 \text{ m}^3 \) while the optimal design solution offers a smaller volume, equal to \( 9.664 \cdot 10^3 \text{ m}^3 \), i.e. a 18.787% reduction.
- The optimal design solution has the transmission ratio for the first stage almost equal to the second stage. That confirms the recommendations found in literature.
- In the optimal design solution the shape of the reducer housing is rather like a “cube” than a “parallelepiped”, which means that the available space will be used more efficiently.
Fig. 2 Optimal and classical design solutions
### Table 3 Classical and optimal design solutions

<table>
<thead>
<tr>
<th>No.</th>
<th>Denotation</th>
<th>Classical solution</th>
<th>Optimal solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Transmission ratio</td>
<td>1.605</td>
<td>2.814</td>
</tr>
<tr>
<td>2</td>
<td>Centre working distance</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(a_{w,i}, \text{[mm]})</td>
<td>100</td>
<td>80</td>
</tr>
<tr>
<td>3</td>
<td>Normal module</td>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>4</td>
<td>Number of teeth of the pinion</td>
<td>z_1 = 38</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>Number of teeth of the wheel</td>
<td>z_2 = 61</td>
<td>76</td>
</tr>
<tr>
<td>6</td>
<td>Standard pitch cylinder helix angle</td>
<td>(\beta_{1}, \text{[deg]})</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>Pinion width</td>
<td>b_{i1}, \text{[mm]}</td>
<td>45</td>
</tr>
<tr>
<td>8</td>
<td>Wheel width</td>
<td>b_{i2}, \text{[mm]}</td>
<td>40</td>
</tr>
<tr>
<td>9</td>
<td>Root diameters</td>
<td>d_{f1}, \text{[mm]}</td>
<td>71.681</td>
</tr>
<tr>
<td></td>
<td></td>
<td>d_{f2}, \text{[mm]}</td>
<td>118.766</td>
</tr>
<tr>
<td>10</td>
<td>Outside diameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(d_{a1}, \text{[mm]})</td>
<td>80.214</td>
<td>47.143</td>
</tr>
<tr>
<td></td>
<td>(d_{a2}, \text{[mm]})</td>
<td>127.319</td>
<td>118.829</td>
</tr>
<tr>
<td>11</td>
<td>Transmission ratio</td>
<td>4.516</td>
<td>2.794</td>
</tr>
<tr>
<td>12</td>
<td>Centre working distance</td>
<td>(a_{w,2}, \text{[mm]})</td>
<td>112</td>
</tr>
<tr>
<td>13</td>
<td>Normal module</td>
<td>(m_{n,2}, \text{[mm]})</td>
<td>1.25</td>
</tr>
<tr>
<td>14</td>
<td>Number of teeth of the pinion</td>
<td>z_3 = 31</td>
<td>34</td>
</tr>
<tr>
<td>15</td>
<td>Number of teeth of the wheel</td>
<td>z_4 = 140</td>
<td>95</td>
</tr>
<tr>
<td>16</td>
<td>Standard pitch cylinder helix angle</td>
<td>(\beta_{2}, \text{[deg]})</td>
<td>15</td>
</tr>
<tr>
<td>17</td>
<td>Pinion width</td>
<td>b_{s1}, \text{[mm]}</td>
<td>68</td>
</tr>
<tr>
<td>18</td>
<td>Wheel width</td>
<td>b_{s2}, \text{[mm]}</td>
<td>63</td>
</tr>
<tr>
<td>19</td>
<td>Root diameters</td>
<td>d_{f3}, \text{[mm]}</td>
<td>39.492</td>
</tr>
<tr>
<td></td>
<td></td>
<td>d_{f4}, \text{[mm]}</td>
<td>178.37</td>
</tr>
<tr>
<td>20</td>
<td>Outside diameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(d_{a3}, \text{[mm]})</td>
<td>45.005</td>
<td>58.243</td>
</tr>
<tr>
<td></td>
<td>(d_{a4}, \text{[mm]})</td>
<td>183.883</td>
<td>147.71</td>
</tr>
<tr>
<td>21</td>
<td>The volume of the reducer housing</td>
<td>(V_{\text{reducer}}, \text{[m}^3])</td>
<td>12.269·10^{-3}</td>
</tr>
</tbody>
</table>

### Acknowledgements

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### References:


